Journal of Sound and Vibration (2001) **242**(3), 507–518
doi:10.1006/jsvi.2000.3357, available online at http://www.idealibrary.com on IDEAL®

A CONTROLLABILITY INDEX FOR OPTIMAL DESIGN OF PIEZOELECTRIC ACTUATORS IN VIBRATION CONTROL OF BEAM STRUCTURES

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(*Received* 4 *August* 1999, *and in final form* 10 *October* 2000)

This paper addresses the controllability aspect in vibration control of beam structures with piezoelectric actuators. First, we model the beam structure with piezoelectric actuators and establish the corresponding state-coupled equation. Based on the state equation, we propose a controllability index to quantify the controllability factor. This index allows one to estimate the amount of control energy by the piezoelectric actuators to the beam for a given control input. So it can be used as an objective function to determine the optimal locations of piezoelectric actuators for vibration control of beam structures. Some numerical simulations are carried out to illustrate the use of this newly proposed controllability index in seeking out the optimal locations of actuators in simply supported and cantilevered beams with one and two actuators. The index is shown to be adequately sensitive for determining the optimal locations of the actuators.

1. INTRODUCTION

Owing to their ability to transform mechanical energy to electrical energy and *vice versa*, piezoelectric materials have been used as both actuators and sensors. The active control of structures using piezoelectric actuators is currently a topic of immense interest in many research fields because [1–3]:

(a) Such smart materials can play the role of distributed actuators and sensors and processor networks which permit a variety of control designs that cannot be used effectively in the case of point controls.
(b) Owing to their small size and lightweight, piezoelectric materials will not add any significant mass to the host structure and therefore they can be used for structural control without significantly modifying the properties of the structure.
(c) These materials have resilient properties against adverse working environments. For example, these materials can withstand the punishing solar radiation in space over extended periods of time.

Surface-bonded piezoelectric actuators are commonly used in structures for control purposes. Various models of beam structures coupled with piezoelectric materials have been proposed by researchers [1, 4–7]. The models differ in the kinematic assumptions and ways of handling the coupling of beams and piezoelectric actuators for dynamic analyses. Ha *et al.* [8] extended the work for active vibration control of composite laminated plates with piezoelectric sensors and actuator layers. A considerable amount of finite element analyses

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and experimental work have been carried out on composite structures with piezoelectric actuators to understand the coupling behaviour [9–16].

From an economic viewpoint, it is desirable to design the piezoelectric actuators for effective vibration control of structures. Optimal design parameters to be determined include the piezoelectric actuators' sizes, locations and the applied voltages. Sungsoo and Librescu [17] investigated the optimal feedback control of structures with respect to the actuators' locations and power consumption. Other research work [18, 19] also investigated the placements of piezoelectric sensors and actuators.

It is well known that misplaced sensors and actuators lead to some problems such as the lack of observability, controllability, and spillover [20]. This paper focuses on the controllability aspect of the piezoelectric actuators bonded on beam structures. We propose in this paper, a way to quantify the controllability factor in the form of an index so that it can be used to determine the optimal locations of piezoelectric actuators. To do this, we first develop a model for the beam structure coupled with piezoelectric actuators and establish the governing state equation for control design. Based on the state equation, we introduce a new controllability index which measures the amount of energy to be provided by the actuators to the beam structure. The controllability index is obtained using a singular value analysis. This same singular value analysis was used by Liu *et al.* [21] to study the controllability of structures with repeated or closely spaced frequencies.

2. MODELLING OF BEAM STRUCTURE WITH PIEZOELECTRIC ACTUATORS

Consider a beam (host) structure with piezoelectric actuators as shown in Figure 1. Adopting the Euler–Bernoulli beam theory, the dynamic equation is given by

$$EI\,\frac{\partial^4 W}{\partial z^4} + \rho A\,\frac{\partial^2 W}{\partial t^2} = f(t) + u(t), \qquad (1)$$

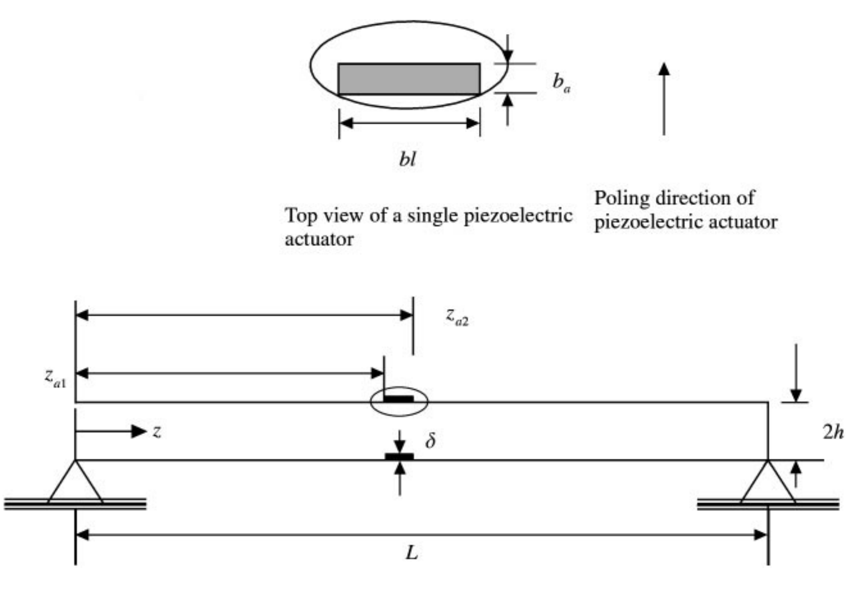

Figure 1. Beam with piezoelectric actuators.

where EI is the flexural rigidity of the beam, ρ the density of the beam, A the cross sectional area of the beam, W the transverse displacement of the beam, t the time, $f(t)$ the external force vector, and $u(t)$ the control input vector provided by the piezoelectric actuators.

For the sake of clarity, we consider one actuator. The poling direction of the piezoelectric material is assumed to be in the direction of the transverse displacement of the beam structure. As a result of supplying a voltage to the piezoelectric actuator, longitudinal strains are induced due to the potential difference between the surfaces of the actuator. The relationship between these longitudinal stresses σ_{11} and strains ε_{11} are given by

$$\sigma_{11} = C_{11}\varepsilon_{11} - e_{31}\frac{v_a}{\delta} = -C_{11}h\frac{\partial^2 W}{\partial z^2} - \frac{e_{31}}{\delta}v_a, \tag{2}$$

where C_{11} is the transformed reduced elastic modulus measured at constant electrical potential, e_{31} the transformed reduced piezoelectric constant, v_a the electrical potential provided to the piezoelectric actuator, h the half thickness of the beam and δ the thickness of piezoelectric actuator.

From simple beam theory, the moment due to the longitudinal stresses is given by

$$M = \frac{\sigma_{11}I_a}{h + \delta/2}, \tag{3}$$

where I_a is the moment inertia of piezoelectric actuator to the mid-plane of beam given by

$$I_a = \frac{b_a\delta^3}{12} + b_a\delta\left(h + \frac{\delta}{2}\right)^2, \tag{4}$$

where b_a is the width of piezoelectric actuator.

By substituting equations (2) and (4) into equation (3), we obtain

$$M = \sigma_{11}b_a\delta\left(h + \frac{\delta}{2}\right) \tag{5}$$

or

$$M = \left[-b_a e_{31}\left(h + \frac{\delta}{2}\right)v_a - b_a C_{11}h\delta\left(h + \frac{\delta}{2}\right)\frac{\partial^2 W}{\partial z^2}\right][\mathrm{H}(z - z_{a2}) - \mathrm{H}(z - z_{a1})], \tag{6}$$

where $\mathrm{H}(\cdot)$ is the Heaviside function, z_{a1}, z_{a2} are the longitudinal coordinates of the end of the piezoelectric actuator as shown in Figure 1.

The controlled system equation (1) may be expressed in terms of the moment M induced by the actuator as follows:

$$EI\frac{\partial^4 W}{\partial z^4} + \rho A\frac{\partial^2 W}{\partial t^2} = f(t) + \frac{\partial^2 M}{\partial z^2}. \tag{7}$$

By substituting equation (6) into equation (7), we obtain

$$EI\frac{\partial^4 W}{\partial z^4} + b_a C_{11}h\delta\left(h + \frac{\delta}{2}\right)\frac{\partial^4 W}{\partial z^4}[\mathrm{H}(z - z_{a2}) - \mathrm{H}(z - z_{a1})] + \rho A\frac{\partial^4 W}{\partial t^2}$$
$$= f(t) - b_a e_{31}\left(h + \frac{\delta}{2}\right)v_a\frac{\partial^2}{\partial z^2}[\mathrm{H}(z - z_{a2}) - \mathrm{H}(z - z_{a1})]. \tag{8}$$

equation (8) represents the coupling model of the beam structure bonded with a piezoelectric actuator. The coupling is evident from the presence of the second term on the left-hand side of equation (8).

The dynamic characteristics (i.e., natural frequencies and mode shapes) of this coupled structure can be determined by assuming harmonic motion and then solving the corresponding eigenvalue problem based on equation (8). Note that the determination of the dynamic characteristics is a prerequisite process for the control design. Usually, the eigenvalue equation (8) is solved by discretizing the beam into a multi-degree freedom system. But, in the following, both the distributed parameter formulation and the lumped parameter formulation are presented for comprehensive treatment of the problem. After obtaining the eigenvalue solution, we can express the response of the system as the summation of mode shapes and mode coordinates in the following form.

For distributed parameter system:

$$W(z,t) = \sum_{i=1}^{n} \phi_i(z) q_i(t). \tag{9}$$

For lumped parameter system:

$$\{W\} = [\phi](\eta), \tag{10}$$

where in equation (9), $\phi_i(t)$ is the mode shape and $q_i(t)$ the mode coordinates. In equation (10) $[\phi]$ is the mode shape matrix, and $\{\eta\}$ the corresponding mode coordinate vector.

In view of equation (9), one may express equation (8) as

$$\begin{aligned}\ddot{q}_i(t) + \omega_i^2 q_i(t) &= f_i(t) - b_a e_{31}(h + 0{\cdot}5\delta) v_a [\phi_i'(z_{a2}) - \phi_i'(z_{a1})] \\ &= f_i(t) + k_a v_a [\phi_i'(z_{a2}) - \phi_i'(z_{a1})],\end{aligned} \tag{11}$$

where the superdot denotes differentiation with respect to time t, and prime denotes differentiation with respect to z.

Now we consider a number of piezoelectric actuators. If this number is k, we can write equation (11) in a state equation form by using $x^{\mathrm{T}} = \{q^{\mathrm{T}}, \dot{q}^{\mathrm{T}}\}$, where $q = \{q_1, q_2 \cdots q_n\}$ and $\dot{q} = \{\dot{q}_1, \dot{q}_2 \cdots \dot{q}_n\}$ and $u = \{v_{a1}, v_{a2}, \cdots, v_{ak}\}^{\mathrm{T}}$, i.e.,

$$\dot{x} = Ax + Bu + f_e(t), \tag{12}$$

where

$$A = \begin{bmatrix} 0 & I \\ -\omega_i^2 & 0 \end{bmatrix}, \tag{13a}$$

$$B = \begin{bmatrix} 0 \\ B_a \end{bmatrix}, \tag{13b}$$

$$B_a = \begin{bmatrix} B_{a1}^1 & B_{a2}^1 & \cdots & B_{ak}^1 \\ B_{a1}^2 & B_{a2}^2 & \cdots & B_{ak}^2 \\ \vdots & \vdots & \vdots & \vdots \\ B_{a1}^n & B_{a2}^n & \cdots & B_{ak}^n \end{bmatrix}, \tag{13c}$$

where $B_{ap}^q = [\phi_q'(z_{ap2}) - \phi_q'(z_{ap1})]$ and $f_e = \{\int \phi_1(z) f(t)\,\mathrm{d}z \cdots \int \phi_n(z) f(t)\,\mathrm{d}z\}^{\mathrm{T}}$.

In this paper, control designs on piezoelectric actuators are based on the state equation (12). In the sequel, the controllability index is introduced which is based on the state equation (12). This index will be used to determine the optimal placements of piezoelectric actuators in vibration control of beam structures.

3. CONTROLLABILITY INDEX

The concept of controllability of system comes from control theory. It is used to determine whether the system can be controlled given the existence of a controller (i.e., the actuator). Based on the form of the ranks of the stiffness matrix A and the control matrix B, one may establish whether the system can be controlled or not. In the following, we attempt to provide a quantitative measure of the controllability in form of a controllability index for the vibration control of beam structures with piezoelectric actuators. This index indicates the amount of control energy supplied by the piezoelectric actuators on the beam structure for a given control input.

From equation (12), we have the control force f_c applied to the system in the vector form of

$$\{f_c\} = [B]\{u\}, \tag{14}$$

where u is electrical potential vector.

Note that

$$\{f_c\}^{\mathrm{T}}\{f_c\} = u^{\mathrm{T}}B^{\mathrm{T}}Bu. \tag{15}$$

By writing $B = MSN^{\mathrm{T}}$ using the singular value analysis where $M \in R^{n\times n}$, $N \in R^{k\times k}$, $M^{\mathrm{T}}M = I$, $NN^{\mathrm{T}} = I$ and

$$S = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_k \\ 0 & 0 & 0 \end{bmatrix}.$$

Here, we assume the number of piezoelectric actuators is less than the number of modes to be controlled.

Equation (15) becomes

$$\{f_c\}^{\mathrm{T}}\{f_c\} = u^{\mathrm{T}}NS^2N^{\mathrm{T}}u, \tag{16}$$

where

$$\mathrm{S}^2 = \begin{bmatrix} \sigma_1^2 & & & \\ & \sigma_2^2 & & \\ & & \ddots & \\ & & & \sigma_k^2 \end{bmatrix}.$$

Let us assume a new input control vector v defined as

$$v = N^{\mathrm{T}}u. \tag{17}$$

Note that $v^2 = u^2$.

It is clear, from the viewpoint of electrical energy, that the new control input v is equivalent to the input vector u. In view of equation (16) and (17), we finally have

$$\{f\}^{\mathrm{T}}\{f\} = v^{\mathrm{T}} S^2 v = \sum_{i=1}^{k} \sigma_i^2 v_i^2. \tag{18}$$

In equation (18), σ_i is referred to as the ith degree of controllability of the system. It is related to the electrical control input v (or u). The magnitude of σ_i is a function of the location and size of the piezoelectric actuators.

Based on σ_i, we now introduce a controllability index Ω

$$\Omega = \prod_{i=1}^{k} \sigma_i. \tag{19}$$

The rational for defining the controllability index in the form by equation (19) is explained below. Recall from equation (16),

$$\{f_c\}^{\mathrm{T}}\{f_c\} = u^{\mathrm{T}} C u, \tag{20}$$

where the matrix $C = NS^2N^{\mathrm{T}}$ and its norm $\|C\|$

$$\|C\| = \|NS^2N^{\mathrm{T}}\| = \|N\|\,\|S^2\|\,\|N^{\mathrm{T}}\| = \|S^2\| = \Omega^2. \tag{21}$$

It follows from equation (20) that for a given control input u, the norm $\|C\| = \Omega^2$ is proportional to the amount of energy $\{f_c\}^{\mathrm{T}}\{f_c\}$. Therefore, the higher the controllability

TABLE 1

Comparison of first five frequencies (in Hz) between pure simply supported beam model and coupled actuator-beam model

	Mode sequence				
Simply supported beam	1	2	3	4	5
Coupled actuator-beam model	523·7	2264·3	5018·9	8462·90	13353·3
Pure beam mode	492·7	1970·5	4413·8	7874·3	12294·6

TABLE 2

Comparison of first five frequencies (in Hz) between pure cantilever beam model and coupled actuator-beam model

	Mode sequence				
Simply supported beam	1	2	3	4	5
Coupled actuator-beam model	203·2	1120·9	3350·5	6769·5	10845·8
Pure beam model	175·6	1100·2	3080·1	6033·8	9969·3

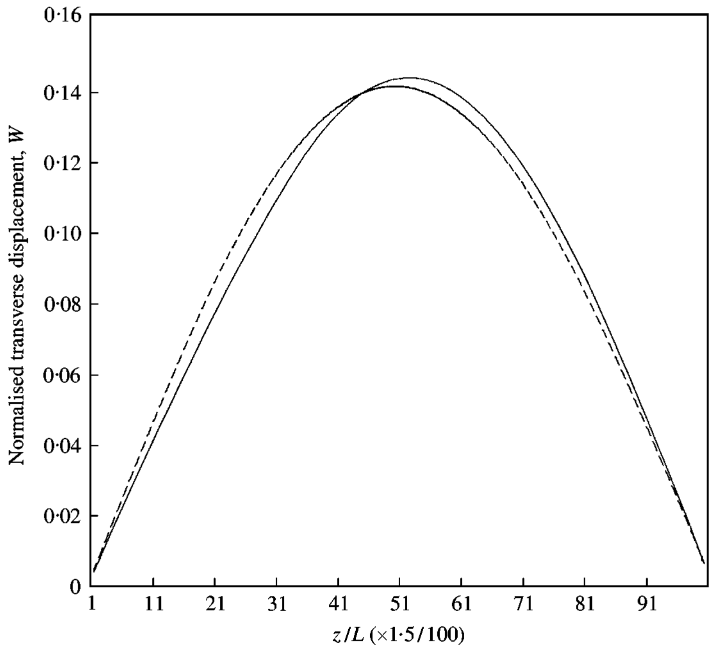

Figure 2. Comparison of the first mode for simply supported beam: ——, piezoelectric actuator coupled beam model; – · · · –, pure beam model.

index Ω, the larger is the control energy provided by the piezoelectric actuators on the beam structure. One may use such an index to determine the optimal locations of piezoelectric actuators where the maximum value of Ω would imply that the control energy is maximized (or used more effectively) for a given control input.

4. NUMERICAL SIMULATIONS

In this section, the foregoing controllability index will be used to obtain the optimal locations of piezoelectric actuators in the vibration control of isotropic beams. All problems solved uses the following material and geometrical parameters.

For beam structure:

$$E = 210 \times 10^9 \,(\mathrm{N/m^2}), \;\; \rho = 7{\cdot}8 \times 10^3 \;\mathrm{kg/m^3}, \mathrm{H} = 0{\cdot}075 \;\mathrm{m}, b = 0{\cdot}075 \;\mathrm{m}, L = 1{\cdot}5 \;\mathrm{m}$$

For piezoelectric actuators.

$$C_{11} = 139 \times 10^9 \;(\mathrm{N/m^2}), \delta = 0{\cdot}01 \;\mathrm{m}, b_h = 0{\cdot}0075 \;\mathrm{m}, bl = 0{\cdot}15 \;\mathrm{m},$$

$$e_{31} = -\;6{\cdot}8 \;(\mathrm{C/m^2}).$$

First, we carry out dynamic analysis of (a) a pure beam model and (b) a beam coupled with piezoelectric actuators model to determine their dynamic characteristics. For the latter coupled actuator-beam model, three actuators are used and their centres are located at 0·25, 0·35, 0·45 m from the left end of the beam. The results obtained will provide information on the effect of piezoelectric actuators on the frequencies and mode shapes of the beam structure.

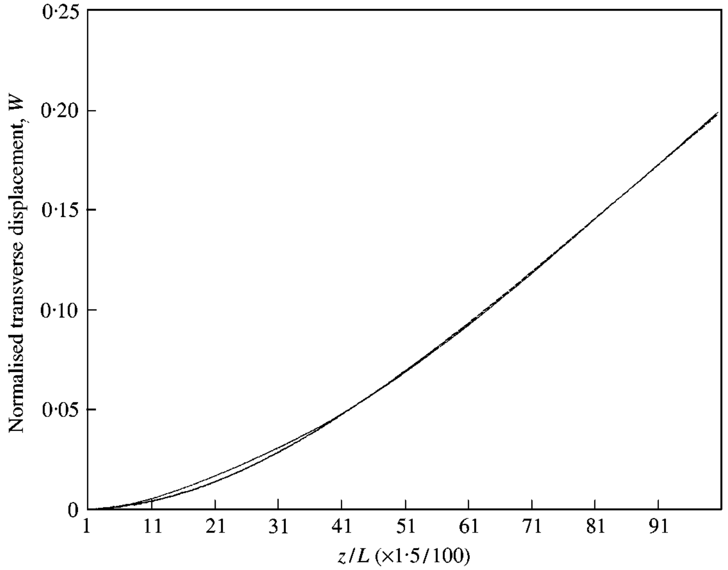

Figure 3. Comparison of first mode for cantilever beam: ——, piezoelectric actuator coupled beam model; –···–, pure beam model.

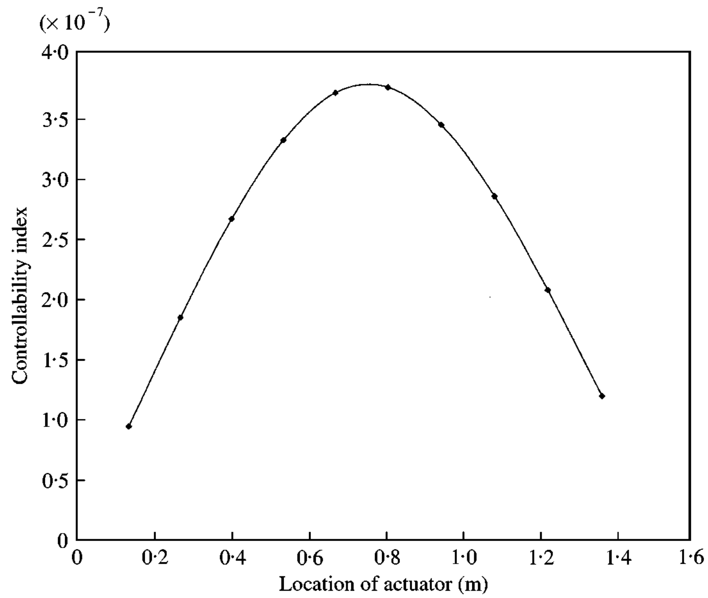

Figure 4. Variation of controllability index with respect to actuator location.

Tables 1 and 2 compare the first several frequencies of the two aforementioned models for the case of simply supported beams and cantilevered beams respectively. Their corresponding first mode shapes are plotted in Figures 2 and 3 respectively. From the comparison studies of the dynamic characteristics, we can see that although the differences of mode shapes are not so obvious, there are clear differences in the frequencies between the two models, i.e. the pure beam model and the coupled actuator-beam model. These

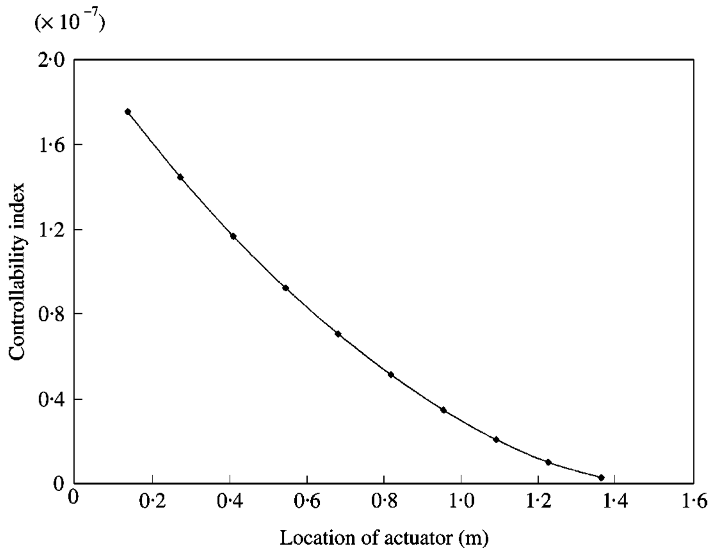

Figure 5. Variation of controllability index with respect to actuator location.

comparison studies showed that such a coupled actuator-beam model is necessary for predicting accurately the dynamic response even though the thickness of the piezoelectric actuators is small.

Now we turn our attention to determine the optimal location of piezoelectric actuators using the controllability index and the coupled actuator-beam model. We first consider simply supported and cantilevered beams with one piezoelectric actuator and focus on just the first mode for vibration control. Figure 4 shows the variation of the controllability index with respect to the location of the piezoelectric actuator bonded to a simply supported beam. It can be seen that the controllability index reaches its highest value when the actuator is placed in the vicinity of the beam's mid-span. Figure 5 shows the relationship between the controllability index and the location of a piezoelectric actuator for the case of the cantilevered beam. For this case, the controllability index is the highest when the actuator is placed near the clamped end. The observations for the two beam cases are expected as the moments are maximum at the mid-span for the case of the simply supported beam and at the clamped end for the case of the cantilevered beam.

If there is a need to control more than one mode shape in the beam structure, more actuators are necessary. The optimal locations of the piezoelectric actuators are not obvious as in the case of using one actuator to control the first mode of vibration. The controllability index becomes extremely useful to determine the optimal locations of the piezoelectric actuators. As an example, consider beams with two piezoelectric actuators. Figure 6 gives the corresponding contour plot of the controllability index $\Omega = \sigma_1 \sigma_2$. It can be seen that the optimal locations of the two piezoelectric actuators are around 0·35 and 1·0 m from the left end of the beam. Further, it can be observed from the contour plot that the controllability index is sensitive to the locations of the two actuators, and thus the index is good for seeking the optimal locations of actuators. Figure 7 gives the corresponding controllability index contour plot for the case of the cantilever beam. The optimal locations of the two piezoelectric actuators are at the clamped end and 0·7 m from the clamped end. As in the case of the simply supported beam, the controllability index is sensitive to the actuator locations.

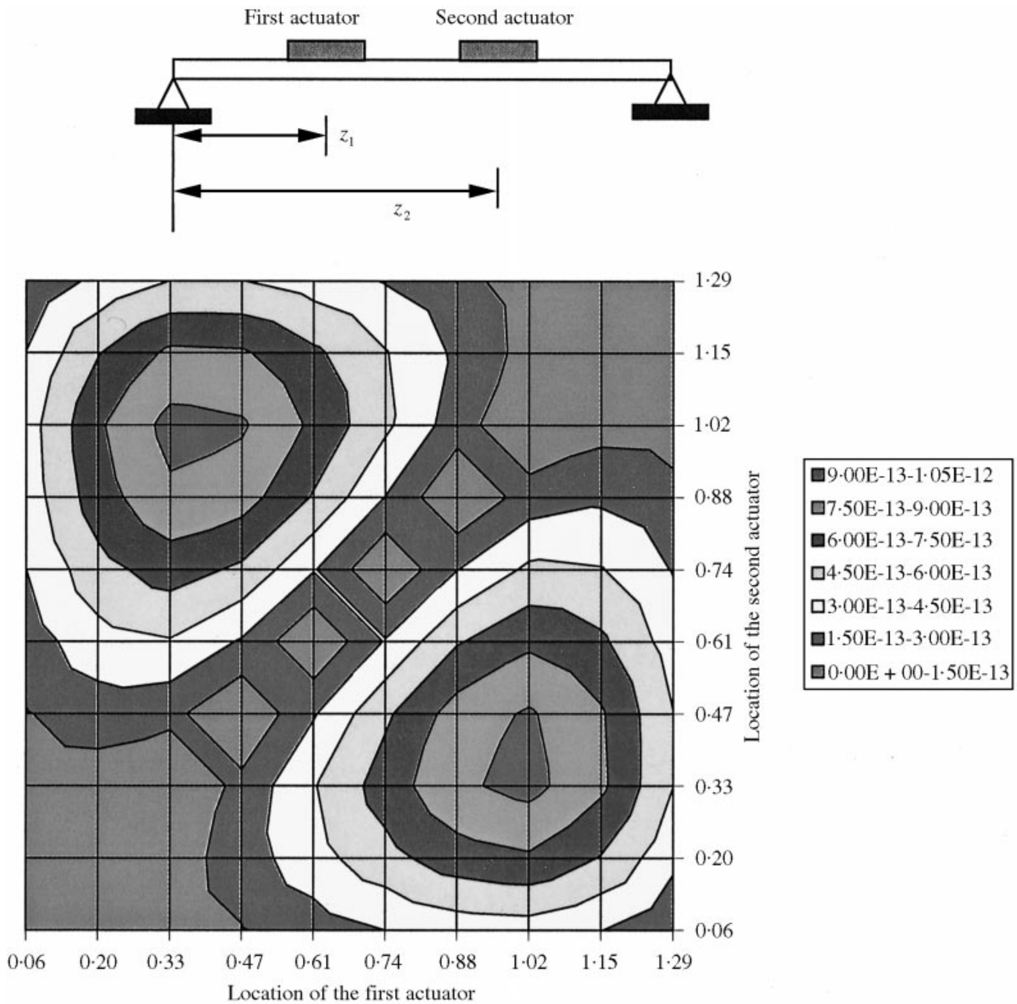

Figure 6. Controllability index for simply supported beam with two actuaros.

5. CONCLUDING REMARKS

This paper dealt with the controllability aspect of the piezoelectric actuators bonded on beam structures. We have proposed a controllability index to quantify the controllability factor. This index can be used to determine the optimal locations of piezoelectric actuators so that the control energy is maximized for a given control input. Based on a developed model for the coupled piezoelectric-beam structure and the new controllability index, the optimal locations of actuators are determined for simply supported and cantilevered beams. The index is shown to be sensitive enough so that the optimal locations of actuators can be determined with ease.

The present work may be extended for vibration control of frames, plates and shells coupled with piezoelectric actuators.

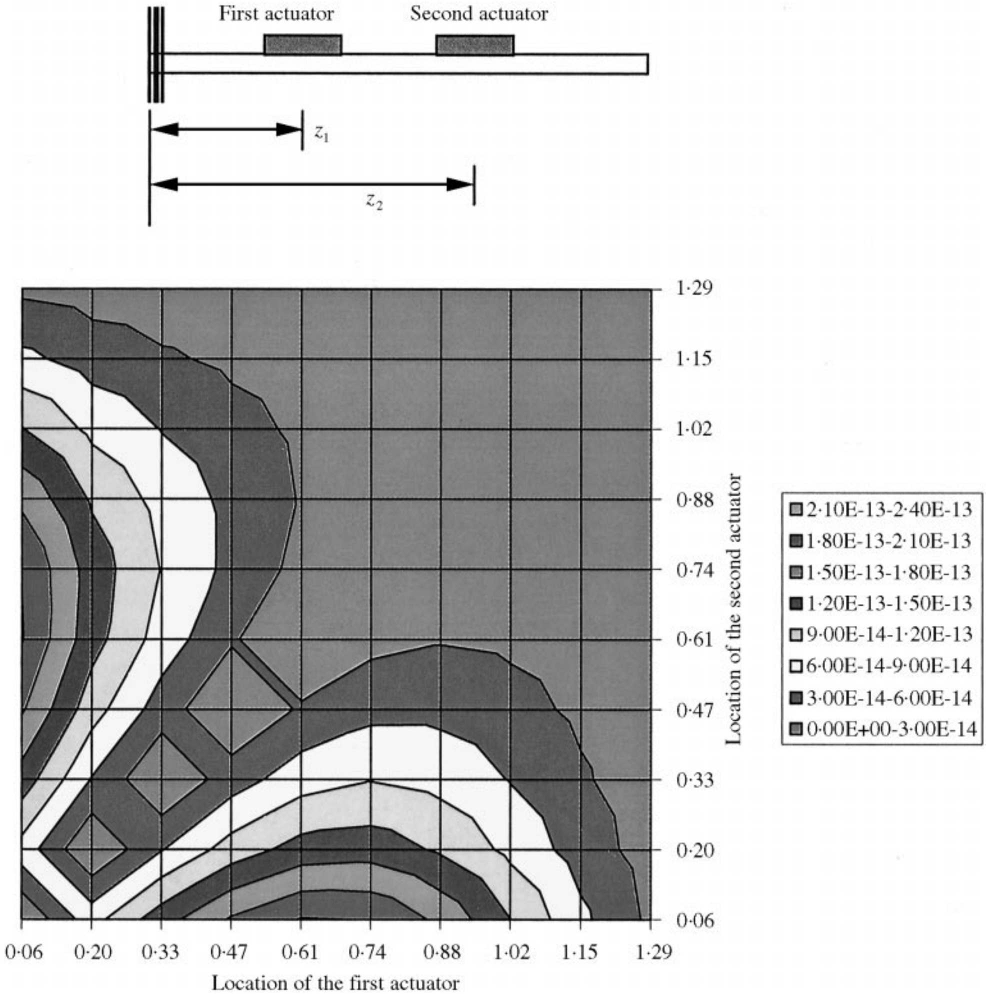

Figure 7. Controllability index for cantilevered beam with two actuators.

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